

# Fully nonlinear and exact perturbations of the Friedmann world model: Non-flat background

**Hyerim Noh**

Korea Astronomy and Space Science Institute, Daejeon 305-348, Republic of Korea

E-mail: [hr@kasi.ac.kr](mailto:hr@kasi.ac.kr)

## **Abstract.**

We extend the fully non-linear and exact cosmological perturbation equations in a Friedmann background universe to include the background curvature. The perturbation equations are presented in a gauge ready form, so any temporal gauge condition can be adopted freely depending on the problem to be solved. We consider the scalar, and vector perturbations without anisotropic stress. As an application, we analyze the equations in the special case of irrotational zero-pressure fluid in the comoving gauge condition. We also present the fully nonlinear formulation for a minimally coupled scalar field.

## **1. Introduction**

The relativistic cosmological perturbation theory is the one of most important theoretical tools to explain the cosmological observations. Bardeen [1] presented a powerful cosmological perturbation formulation in which the temporal gauge conditions can be applied freely depending on the characters of the problem to solve [1]. Bardeen's linear cosmological perturbation formulation was extended in [2, 3, 4] and to the second order [5]. In the recent work [6], we have extended Bardeen's formulation to the fully non-linear cosmological perturbations in the flat Friedmann universe. In [6] the equations are presented in the gauge ready form, so the temporal gauge is not fixed. We assumed the flat Friedmann background universe. We considered the scalar, and vector perturbations of a fluid without anisotropic stress. As is explained in [6], the transverse and tracefree tensor perturbations are not considered because of the technical difficulties. Instead, the tensor perturbation can always be handled to the nonlinear order perturbatively.

In this work, as one of the extensions we consider the fully non-linear perturbations in Friedmann background universe with the general background curvature.

In Section 2 we summarize the metric and the energy momentum tensor convention and the gauge conditions. In Section 3 we present the exact and fully nonlinear perturbation equations. In Section 4, as an application we show the case in the comoving gauge. We present the closed form of second-order differential equation for the perturbed

energy density  $\delta$ . Especially, the zero-pressure case was analyzed in detail. In Section 5 we present the fully nonlinear perturbation equations for the minimally coupled scalar field in a non-flat background model. In Section 6 the discussion is given. In the Appendix, the details useful for deriving the fully nonlinear perturbation equations are shown.

## 2. Convention and gauge condition

In perturbed Friedmann universe with a background curvature the metric is given as

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a^2 (\beta_{,i} + B_i^{(v)}) d\eta dx^i + a^2 \left[ (1 + 2\varphi) g_{ij}^{(3)} + 2\gamma_{i|j} + C_{i|j}^{(v)} + C_{j|i}^{(v)} + 2C_{ij}^{(t)} \right] dx^i dx^j, \quad (1)$$

where  $a(\eta)$  is the cosmic scale factor. We assume  $B_i^{(v)}|_i \equiv 0 \equiv C_i^{(v)}|_i$ , and  $C_i^{(t)} = 0 = C_i^{(t)j}|_j$  with indices of  $B_i^{(v)}$ ,  $C_i^{(v)}$  and  $C_{ij}^{(t)}$  raised and lowered by  $g_{ij}^{(3)}$  as the metric;  $g_{ij}^{(3)}$  becomes  $\delta_{ij}$  in a flat background; indices  $(v)$  and  $(t)$  indicate the vector- and tensor-type perturbations, respectively; a vertical bar indicates the covariant derivative based on the  $g_{ij}^{(3)}$  as the metric tensor. Indices  $a, b, \dots$  indicate the spacetime indices, and  $i, j, \dots$  indicate the spatial ones. The metric  $g_{ij}^{(3)}$  is the background comoving three-space part of the Robertson-Walker metric, and is given as

$$\begin{aligned} g_{ij}^{(3)} dx^i dx^j &= \frac{dr^2}{1 - \bar{K}r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \\ &= \frac{1}{(1 + \frac{\bar{K}}{4} \bar{r}^2)^2} (dx^2 + dy^2 + dz^2) \\ &= d\bar{\chi}^2 + \left[ \frac{1}{\sqrt{\bar{K}}} \sin(\sqrt{\bar{K}} \bar{\chi}) \right]^2 (d\theta^2 + \sin^2 \theta d\phi^2), \end{aligned} \quad (2)$$

where

$$r \equiv \frac{\bar{r}}{1 + \frac{\bar{K}}{4} \bar{r}^2}, \quad \bar{r} \equiv \sqrt{x^2 + y^2 + z^2}, \quad \bar{\chi} \equiv \int^r \frac{dr}{\sqrt{1 - \bar{K}r^2}}. \quad (3)$$

The background curvature  $\bar{K}$  is defined in equation (64).

We decompose the spatial vector into longitudinal and transverse parts as  $B_i = \beta_{,i} + B_i^{(v)}$ , and a symmetric spatial tensor into longitudinal, trace, transverse, and tracefree-transverse parts as  $C_{ij} = \varphi g_{ij}^{(3)} + \gamma_{i|j} + \frac{1}{2}(C_{i|j}^{(v)} + C_{j|i}^{(v)}) + C_{ij}^{(t)}$  [7]. All spatial indices are raised and lowered by  $g_{ij}^{(3)}$  as the metric. The transverse part corresponds to the vector-type perturbation, and the tracefree-transverse part corresponds to the tensor-type perturbation. The longitudinal and trace parts correspond to the scalar-type perturbation. In the homogeneous-isotropic background the three types of perturbations decouple from each other, only to the linear order. In the nonlinear order we have couplings among the scalar-, vector- and tensor-types of perturbations. As in [6], here we consider only the scalar- and vector-type perturbations.

As the spatial gauge condition we choose

$$\gamma \equiv 0 \equiv C_i^{(v)}. \quad (4)$$

This is the only spatial gauge condition which does not leave the gauge mode and allows the derivation of the fully nonlinear perturbation equations; see Section 2 of [6]. Then, our metric becomes

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a\chi_i d\eta dx^i + a^2 (1 + 2\varphi) g_{ij}^{(3)} dx^i dx^j, \quad (5)$$

where

$$\chi_i = a \left( \beta_{,i} + B_i^{(v)} \right) \equiv \chi_{,i} + \chi_i^{(v)}. \quad (6)$$

Considering an ideal fluid and choosing the energy frame we have the energy momentum tensor [8, 9, 10, 11].

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} (\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab}), \quad (7)$$

where  $\tilde{\mu}$  and  $\tilde{p}$  are the energy density and the pressure, respectively,  $\tilde{u}_a$  is the normalized fluid four-vector with  $\tilde{u}^a \tilde{u}_a \equiv -1$ ; tildes indicate the covariant quantities. We define

$$\tilde{\mu} \equiv \mu + \delta\mu \equiv \mu (1 + \delta), \quad \tilde{p} \equiv p + \delta p, \quad \tilde{u}_i \equiv \frac{a}{c} \hat{\gamma} \hat{v}_i, \quad \hat{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{\hat{v}^k \hat{v}_k}{c^2(1+2\varphi)}}}, \quad (8)$$

where  $\mu$  and  $p$  are the background energy density and pressure, respectively;  $\hat{\gamma}$  is the Lorentz factor.

We decompose

$$\hat{v}_i \equiv -\hat{v}_{,i} + \hat{v}_i^{(v)}, \quad (9)$$

with  $\hat{v}^{(v)i}|_i \equiv 0$ ;  $\hat{v}_i$  and  $\hat{v}_i^{(v)}$  are raised and lowered by  $g_{ij}^{(3)}$  as the metric. The perturbed metric and fluid quantities could have arbitrary amplitude.

Our fully nonlinear equations are arranged without fixing the temporal gauge conditions which we call the “gauge ready” methods. As Bardeen introduced [1], the gauge ready method has the advantage in the sense that any gauge conditions can be adopted depending on the problem at hand. We have several temporal gauge conditions available. The fundamental gauge conditions are the comoving gauge ( $\hat{v} = 0$ ), the zero-shear gauge ( $\chi = 0$ ), the uniform-curvature gauge ( $\varphi = 0$ ), the uniform expansion gauge ( $\kappa = 0$ ), the uniform-density gauge ( $\delta = 0$ ), and the synchronous gauge ( $\alpha = 0$ ); for  $\kappa$ , see later. Except for the synchronous gauge, all above gauge conditions remove the gauge mode completely, and the variables in those gauge conditions are gauge invariant. The detailed gauge issues to the nonlinear order are explained in Section 2 of [6].

### 3. Exact and fully nonlinear perturbation equations in a gauge-ready form

Based on the ADM (Arnowitt-Deser-Misner) and the covariant formulation [12], [8]-[10] we can derive the fully non-linear perturbation equations in a non-flat Friedmann background universe; the ADM and the covariant set of equations are presented in the Appendix A and C of [6], and the quantities useful for derivation are presented in the Appendix. An overdot indicates the covariant derivative based on  $t$  with  $cdt \equiv ad\eta$ .

Definition of  $\kappa$ :

$$\kappa + 3H \left( \frac{1}{\mathcal{N}} - 1 \right) + \frac{1}{\mathcal{N}(1+2\varphi)} \left[ 3\dot{\varphi} + \frac{c}{a^2} \left( \chi^k{}_{|k} + \frac{\chi^k \varphi_{,k}}{1+2\varphi} \right) \right] = 0. \quad (10)$$

ADM energy constraint:

$$\begin{aligned} & -\frac{3}{2} \left( H^2 - \frac{8\pi G}{3c^2} \tilde{\mu} + \frac{\overline{K}c^2}{a^2(1+2\varphi)} - \frac{\Lambda c^2}{3} \right) + H\kappa + \frac{c^2 \Delta \varphi}{a^2(1+2\varphi)^2} \\ & = \frac{1}{6} \kappa^2 - \frac{4\pi G}{c^2} (\tilde{\mu} + \tilde{p}) (\hat{\gamma}^2 - 1) + \frac{3}{2} \frac{c^2 \varphi^{[i} \varphi_{,i}}{a^2(1+2\varphi)^3} - \frac{c^2}{4} \overline{K}_j^i \overline{K}_i^j. \end{aligned} \quad (11)$$

ADM momentum constraint:

$$\begin{aligned} & \frac{2}{3} \kappa_{,i} + \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left[ \frac{1}{2} (\chi_i{}^{[k}{}_{|k} + \chi^k{}_{|ik}) - \frac{1}{3} \chi^k{}_{|ki} \right] + \frac{8\pi G}{c^4} (\tilde{\mu} + \tilde{p}) a \hat{\gamma}^2 \hat{v}_i \\ & = \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left\{ \left( \frac{\mathcal{N}_{,j}}{\mathcal{N}} - \frac{\varphi_{,j}}{1+2\varphi} \right) \left[ \frac{1}{2} (\chi^j{}_{|i} + \chi_i{}^{[j}{}_{|j}) - \frac{1}{3} \delta_i^j \chi^k{}_{|k} \right] \right. \\ & \quad \left. - \frac{\varphi^{[j}}{(1+2\varphi)^2} (\chi_i \varphi_{,j} + \frac{1}{3} \chi_j \varphi_{,i}) + \frac{\mathcal{N}}{1+2\varphi} \left[ \frac{1}{\mathcal{N}} (\chi^j \varphi_{,i} + \chi_i \varphi^{[j}{}_{|j} - \frac{2}{3} \delta_i^j \chi^k \varphi_{,k}) \right]_{|j} \right\}. \end{aligned} \quad (12)$$

Trace of ADM propagation:

$$\begin{aligned} & -3 \left[ \frac{1}{\mathcal{N}} \dot{H} + H^2 + \frac{4\pi G}{3c^2} (\tilde{\mu} + 3\tilde{p}) - \frac{\Lambda c^2}{3} \right] + \frac{\dot{\kappa}}{\mathcal{N}} + 2H\kappa + \frac{c^2 \mathcal{N}^{[k}{}_{|k}}{a^2 \mathcal{N}(1+2\varphi)} \\ & = \frac{1}{3} \kappa^2 + \frac{8\pi G}{c^2} (\tilde{\mu} + \tilde{p}) (\hat{\gamma}^2 - 1) - \frac{c}{a^2 \mathcal{N}(1+2\varphi)} (\chi^i \kappa_{,i} + c \frac{\varphi^{[i} \mathcal{N}_{,i}}{1+2\varphi}) + c^2 \overline{K}_j^i \overline{K}_i^j. \end{aligned} \quad (13)$$

Tracefree ADM propagation:

$$\begin{aligned} & \left( \frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + 3H - \kappa \right) \left\{ \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left[ \frac{1}{2} (\chi^i{}_{|j} + \chi_j{}^{[i}{}_{|i}) - \frac{1}{3} \delta_j^i \chi^k{}_{|k} \right. \right. \\ & \quad \left. \left. - \frac{1}{1+2\varphi} (\chi^i \varphi_{,j} + \chi_j \varphi^{[i}{}_{|i} - \frac{2}{3} \delta_j^i \chi^k \varphi_{,k}) \right] \right\} + \frac{c \chi^k}{a^2 \mathcal{N}(1+2\varphi)} \\ & \times \left\{ \frac{c}{a^2 \mathcal{N}(1+2\varphi)} \left[ \frac{1}{2} (\chi^i{}_{|j} + \chi_j{}^{[i}{}_{|i}) - \frac{1}{3} \delta_j^i \chi^l{}_{|l} - \frac{1}{1+2\varphi} (\chi^i \varphi_{,j} + \chi_j \varphi^{[i}{}_{|i} - \frac{2}{3} \delta_j^i \chi^l \varphi_{,l}) \right] \right\}_{|k} \\ & - \frac{c^2}{a^2(1+2\varphi)} \left[ \frac{1}{1+2\varphi} (\varphi^{[i}{}_{|j} - \frac{1}{3} \delta_j^i \varphi^{[k}{}_{|k}) + \frac{1}{\mathcal{N}} (\mathcal{N}^{[i}{}_{|j} - \frac{1}{3} \delta_j^i \mathcal{N}^{[k}{}_{|k}) \right] \\ & = \frac{8\pi G}{c^2} (\tilde{\mu} + \tilde{p}) \left[ \frac{\hat{\gamma}^2 \hat{v}^i \hat{v}_j}{c^2(1+2\varphi)} - \frac{1}{3} \delta_j^i (\hat{\gamma}^2 - 1) \right] + \frac{c^2}{a^4 \mathcal{N}^2(1+2\varphi)^2} \left[ \frac{1}{2} (\chi^i{}_{|k} \chi_j{}^{[k}{}_{|k} - \chi_{k|j} \chi^k{}_{|i}) \right. \\ & \quad \left. + \frac{1}{1+2\varphi} (\chi^{[k}{}_{|i} \chi_k \varphi_{,j} - \chi^{[i}{}_{|k} \chi_j \varphi_{,k} + \chi_{k|j} \chi^k \varphi^{[i}{}_{|i} - \chi_{j|k} \chi^i \varphi^{[k}{}_{|k}) \right. \\ & \quad \left. + \frac{2}{(1+2\varphi)^2} (\chi^i \chi_j \varphi^{[k}{}_{|k} \varphi_{,k} - \chi^k \chi_k \varphi^{[i}{}_{|i} \varphi_{,j}) \right] - \frac{c^2}{a^2(1+2\varphi)^2} \left[ \frac{3}{1+2\varphi} \right. \\ & \quad \left. \times (\varphi^{[i}{}_{|j} \varphi_{,j} - \frac{1}{3} \delta_j^i \varphi^{[k}{}_{|k} \varphi_{,k}) + \frac{1}{\mathcal{N}} (\varphi^{[i} \mathcal{N}_{,j} + \varphi_{,j} \mathcal{N}^{[i}{}_{|i} - \frac{2}{3} \delta_j^i \varphi^{[k}{}_{|k} \mathcal{N}_{,k}) \right]. \end{aligned} \quad (14)$$

ADM energy conservation:

$$\frac{1}{\mathcal{N}} [\tilde{\mu} + (\tilde{\mu} + \tilde{p})(\hat{\gamma}^2 - 1)] + \frac{c}{a^2 \mathcal{N}} \frac{\chi^i}{1+2\varphi} [\tilde{\mu} + (\tilde{\mu} + \tilde{p})(\hat{\gamma}^2 - 1)]_{|i}$$

$$\begin{aligned}
& +(\tilde{\mu} + \tilde{p})(3H - \kappa)\frac{1}{3}(4\hat{\gamma}^2 - 1) + \left(\frac{\tilde{\mu} + \tilde{p}}{a(1+2\varphi)}\hat{\gamma}^2\hat{v}^i\right)_{|i} + \left(\frac{3\varphi_{,i}}{1+2\varphi} + 2\frac{\mathcal{N}_{,i}}{\mathcal{N}}\right)\frac{\tilde{\mu} + \tilde{p}}{a(1+2\varphi)}\hat{\gamma}^2\hat{v}^i \\
& = -\frac{\hat{\gamma}^2(\tilde{\mu} + \tilde{p})}{ca^2\mathcal{N}(1+2\varphi)^2}\left[\chi^{i|j}\hat{v}_i\hat{v}_j - \frac{1}{3}\chi^i{}_{|j}\hat{v}^j\hat{v}_i - \frac{2}{1+2\varphi}\left(\hat{v}^i\hat{v}^j\chi_{i\varphi,j} - \frac{1}{3}\hat{v}^i\hat{v}_i\chi^j\varphi_{,j}\right)\right]. \quad (15)
\end{aligned}$$

ADM momentum conservation:

$$\begin{aligned}
& \left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t} + 3H - \kappa\right)\left[a(\tilde{\mu} + \tilde{p})\hat{\gamma}^2\hat{v}_i\right] + \frac{c}{a^2\mathcal{N}}\frac{\chi^j}{(1+2\varphi)}\left[a(\tilde{\mu} + \tilde{p})\hat{\gamma}^2\hat{v}_i\right]_{|j} + c^2\tilde{p}_{,i} + c^2(\tilde{\mu} + \tilde{p})\frac{\mathcal{N}_{,i}}{\mathcal{N}} \\
& = -\left[(\tilde{\mu} + \tilde{p})\frac{\hat{\gamma}^2\hat{v}^j\hat{v}_i}{1+2\varphi}\right]_{|j} - \frac{c}{a\mathcal{N}}\left(\frac{\chi^j}{1+2\varphi}\right)_{|i}(\tilde{\mu} + \tilde{p})\hat{\gamma}^2\hat{v}_j \\
& - \frac{\tilde{\mu} + \tilde{p}}{1+2\varphi}\hat{\gamma}^2\hat{v}^j\left[\frac{1}{1+2\varphi}(3\hat{v}_i\varphi_{,j} - \hat{v}_j\varphi_{,i}) + \frac{1}{\mathcal{N}}(\hat{v}_i\mathcal{N}_{,j} + \hat{v}_j\mathcal{N}_{,i})\right]. \quad (16)
\end{aligned}$$

Instead of the ADM-energy and momentum conservation equations, we can use the covariant energy and the momentum conservation equations; see the Appendix C in [6].

Covariant energy conservation:

$$\begin{aligned}
& \left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^i + \frac{c}{a}\chi^i\right)\nabla_i\right]\tilde{\mu} + (\tilde{\mu} + \tilde{p})\left\{(3H - \kappa)\mathcal{N} + \frac{(\mathcal{N}\hat{v}^i)_{|i}}{a(1+2\varphi)}\right. \\
& \left. + \frac{\mathcal{N}\hat{v}^i\varphi_{,i}}{a(1+2\varphi)^2} + \frac{1}{\hat{\gamma}}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^i + \frac{c}{a}\chi^i\right)\nabla_i\right]\hat{\gamma}\right\} = 0. \quad (17)
\end{aligned}$$

Covariant momentum conservation:

$$\begin{aligned}
& \frac{\partial}{\partial t}(a\hat{\gamma}\hat{v}_i) + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k + \frac{c}{a}\chi^k\right)(a\hat{\gamma}\hat{v}_i)_{|k} \\
& + \frac{1}{\tilde{\mu} + \tilde{p}}\left\{c^2\frac{\mathcal{N}}{\hat{\gamma}}\tilde{p}_{,i} + a\hat{\gamma}\hat{v}_i\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k + \frac{c}{a}\chi^k\right)\nabla_k\right]\tilde{p}\right\} \\
& + c^2\hat{\gamma}\mathcal{N}_{,i} + \frac{1 - \hat{\gamma}^2}{\hat{\gamma}}\frac{c^2\mathcal{N}\varphi_{,i}}{1+2\varphi} + \frac{c}{a}\hat{\gamma}\hat{v}^k\left(\frac{\chi_k}{1+2\varphi}\right)_{|i} = 0. \quad (18)
\end{aligned}$$

The equation (10) is derived from the trace of the extrinsic curvature as  $K \equiv -3H + \kappa$ ;  $\mathcal{N}$  is related to the lapse function in equation (60), and  $\overline{K}_j^i$  is the tracefree part of extrinsic curvature in equation (66).  $\mathcal{N}$  and  $\overline{K}_j^i\overline{K}_i^j$  are given as

$$\begin{aligned}
\mathcal{N} &= \sqrt{1 + 2\alpha + \frac{\chi^k\chi_k}{a^2(1+2\varphi)}}, \\
\overline{K}_j^i\overline{K}_i^j &= \frac{1}{a^4\mathcal{N}^2(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i|j}\left(\chi_{i|j} + \chi_{j|i}\right) - \frac{1}{3}\chi^i{}_{|i}\chi^j{}_{|j}\right. \\
&\quad - \frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{j|j}\left(\chi_{i|j} + \chi_{j|i}\right) - \frac{1}{3}\chi^i{}_{|i}\chi^j\varphi_{,j}\right] \\
&\quad \left. + \frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{j|j}\varphi_{,j} + \frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}. \quad (19)
\end{aligned}$$

Now, we have the complete set of the fully nonlinear perturbations in (10)-(18) in a non-flat background universe.

The dimensions are,

$$\begin{aligned} [a] = [\tilde{g}_{ab}] = [\tilde{u}_a] = [\alpha] = [\varphi] = [\chi^i] = [v^i/c] = [\hat{v}^i/c] &= 1, \\ [v/c] = L, \quad [x^i] = [cdt] \equiv [ad\eta] = L, \quad [\chi] = L, \quad [\kappa] = T^{-1}, \\ [\tilde{T}_{ab}] = [\tilde{\mu}] = [\tilde{\rho}c^2] = [\tilde{p}], \quad [G\tilde{\rho}] = T^{-2}, \quad [\overline{K}] = L^{-2}, \quad [\Lambda] = L^{-2}. \end{aligned} \quad (20)$$

#### 4. Analysis in the Comoving gauge

We see that the background curvature term explicitly appears in the ADM energy constraint equation only. The effect of background curvature appears in the ADM momentum constraint equation as well: see equations (11) and (12).

In order to see the effects of the background curvature in detail, in this section we consider the scalar-type perturbations, so  $\hat{v}_i = -\hat{v}_{,i}$  and  $\chi_i = \chi_{,i}$ . We take the comoving gauge

$$\hat{v} \equiv 0, \quad (21)$$

thus  $\hat{v}_i = 0$ . From equation (16) or (18) we have

$$\tilde{p}_{,i} = -(\tilde{\mu} + \tilde{p}) \frac{\mathcal{N}_{,i}}{\mathcal{N}}. \quad (22)$$

Equations (10)-(18) are the complete set of equations for the variables  $\delta$ ,  $\kappa$ ,  $\varphi$ ,  $\chi$  and  $\alpha$ . Since the comoving gauge completely fixes the gauge modes, all perturbation variables in the comoving gauge are gauge invariant even to the nonlinear order: see Section 2 in [6].

Subtracting the background equation, equations (15) or (17) gives

$$\kappa = \frac{1}{\tilde{\mu} + \tilde{p}} \frac{1}{\mathcal{N}} \left( \frac{\partial}{\partial t} + \frac{c\chi^{[i}}{a^2(1+2\varphi)} \nabla_i \right) \tilde{\mu} - \frac{\dot{\mu}}{\mu + p}. \quad (23)$$

Equation (13) gives

$$\begin{aligned} \frac{1}{\mathcal{N}} \dot{\kappa} + 2H\kappa - 3 \left( \frac{1}{\mathcal{N}} - 1 \right) \dot{H} - \frac{4\pi G}{c^2} (\delta\mu + 3\delta p) + \frac{c^2 \mathcal{N}^{[i}}{a^2 \mathcal{N} (1+2\varphi)} \\ = \frac{1}{3} \kappa^2 - \frac{c}{a^2 \mathcal{N} (1+2\varphi)} \left( \chi^{[i} \kappa_{,i} + c \frac{\varphi^{[i} \mathcal{N}_{,i}}{1+2\varphi} \right) + c^2 \overline{K}_j^i \overline{K}_i^j. \end{aligned} \quad (24)$$

Combining these we have a second-order differential equation for  $\tilde{\mu}$ .

##### 4.1. Zero-pressure case

Now, we assume the pressure is negligible, then  $\mathcal{N}_{,i} = 0$  from equation (22). Thus, we have  $\mathcal{N} = 1$  and

$$\alpha = -\frac{1}{2} \frac{\chi^{[k} \chi_{,k}}{a^2 (1+2\varphi)}. \quad (25)$$

Notice that even in the zero-pressure case, with our choice of the spatial gauge condition  $\gamma = 0$ , to the nonlinear order the comoving gauge ( $\hat{v} = 0$ ) does not imply the synchronous gauge ( $\alpha = 0$ ) [13].

Using equations (13), (15), (11) and (12) we have the complete set of equations for  $\delta$  and  $\kappa$ . Equation (23) and (24), respectively, give

$$\dot{\delta} - \kappa = \delta\kappa - \frac{c\chi^{[i}\delta_{,i}}{a^2(1+2\varphi)}, \quad (26)$$

$$\dot{\kappa} + 2H\kappa - \frac{4\pi G}{c^2}\delta\mu = \frac{1}{3}\kappa^2 - \frac{c\chi^{[i}\kappa_{,i}}{a^2(1+2\varphi)} + c^2\overline{K}_j^i\overline{K}_i^j. \quad (27)$$

From these we have

$$\ddot{\delta} + 2H\dot{\delta} - \frac{4\pi G\mu}{c^2}\delta = \frac{1}{a^2}(a^2\kappa\delta)' - \frac{c}{a^2}\left(\frac{\chi^{[i}\delta_{,i}}{1+2\varphi}\right)' + \frac{1}{3}\kappa^2 - \frac{c\chi^{[i}\kappa_{,i}}{a^2(1+2\varphi)} + c^2\overline{K}_j^i\overline{K}_i^j. \quad (28)$$

The variables  $\chi$  and  $\varphi$  are obtained from equation (12) and (11), respectively

$$\begin{aligned} \kappa_{,i} + \frac{c}{a^2(1+2\varphi)}\left[(\Delta + 3\overline{K})\chi\right]_{,i} \\ = \frac{c}{a^2(1+2\varphi)^2}\left[\frac{3}{2}\left(\chi^{[j}\varphi_{,i|j} + \chi_{,i|j}\varphi^{[j} - \frac{2}{3}\chi^{[j}{}_i\varphi_{,j} - \frac{2}{3}\chi^{[j}\varphi_{,j|i}\right) \right. \\ \left. + 2\chi^{[j}{}_j\varphi_{,i} - \frac{3}{2}\varphi_{,j}\chi^{[j}{}_i + \frac{3}{2}\chi_{,i}\varphi^{[j}{}_j - \frac{3}{2}\frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,j} + \frac{1}{3}\chi_{,j}\varphi_{,i}\right)\varphi^{[j}{}_j\right], \end{aligned} \quad (29)$$

$$\begin{aligned} -\frac{3}{2}\left(H^2 - \frac{8\pi G}{3c^2}\mu + \frac{c^2\overline{K}}{a^2(1+2\varphi)} - \frac{\Lambda c^2}{3}\right) + H\kappa + \frac{4\pi G}{c^2}\mu\delta + \frac{c^2\varphi^{[i}{}_{,i}}{a^2(1+2\varphi)^2} \\ = \frac{1}{6}\kappa^2 + \frac{3}{2}\frac{c^2\varphi^{[i}\varphi_{,i}}{a^2(1+2\varphi)^3} - \frac{c^2}{4}\overline{K}_j^i\overline{K}_i^j. \end{aligned} \quad (30)$$

To the linear order, using equations (10) and (29) we have

$$\dot{\varphi} = \frac{\overline{K}c}{a^2}\chi. \quad (31)$$

Thus, in the presence of the background curvature,  $\varphi$  in the comoving gauge is not conserved even to the linear order, while in a flat background, we have  $\dot{\varphi} = 0$ .

#### 4.2. Pure relativistic corrections to fully nonlinear order

In order to see the pure Einstein's gravity corrections, we arrange Equations (26), (27) and (29) as

$$\dot{\delta} - \kappa - \delta\kappa + \frac{c}{a^2}\chi^{[i}\delta_{,i} = \frac{2c\varphi\chi^{[i}\delta_{,i}}{a^2(1+2\varphi)}, \quad (32)$$

$$\begin{aligned} \dot{\kappa} + 2H\kappa - \frac{4\pi G}{c^2}\delta\mu - \frac{1}{3}\kappa^2 + \frac{c}{a^2}\chi^{[i}\kappa_{,i} - \frac{c^2}{a^4}\left[\chi^{[ij}\chi_{,i|j} - \frac{1}{3}(\Delta\chi)^2\right] \\ = \frac{2c\varphi\chi^{[i}\kappa_{,i}}{a^2(1+2\varphi)} - \frac{4c^2\varphi(1+\varphi)}{a^4(1+2\varphi)^2}\left[\chi^{[ij}\chi_{,i|j} - \frac{1}{3}(\Delta\chi)^2\right] \\ + \frac{2c^2}{a^4(1+2\varphi)^3}\left\{\frac{2}{3}(\Delta\chi)\chi^{[i}\varphi_{,i} - 2\chi^{[ij}\chi_{,i}\varphi_{,j} + \frac{1}{1+2\varphi}\left[\frac{1}{3}\left(\chi^{[i}\varphi_{,i}\right)^2 + \chi^{[i}\chi_{,i}\varphi^{[j}{}_j\varphi_{,j}\right]\right\} \quad (33) \\ \kappa_{,i} + \frac{c}{a^2}\left[(\Delta + 3\overline{K})\chi\right]_{,i} = \frac{2c\varphi}{a^2(1+2\varphi)}\left[(\Delta + 3\overline{K})\chi\right]_{,i} \end{aligned}$$

$$\begin{aligned}
& + \frac{c}{a^2(1+2\varphi)^2} \left[ \frac{3}{2} \left( \chi^{[j} \varphi_{,i|j} + \chi_{,i|j} \varphi^{j]} - \frac{2}{3} \chi^{[j}{}_{i} \varphi_{,j} - \frac{2}{3} \chi^{[j} \varphi_{,j|i} \right) + 2 \chi^{[j}{}_{j} \varphi_{,i} \right. \\
& \left. - \frac{3}{2} \varphi_{,j} \chi^{[j}{}_{i} + \frac{3}{2} \chi_{,i} \varphi^{j]}{}_{j} - \frac{3}{2} \frac{1}{1+2\varphi} \left( \chi_{,i} \varphi_{,j} + \frac{1}{3} \chi_{,j} \varphi_{,i} \right) \varphi^{j]} \right]. \tag{34}
\end{aligned}$$

The pure relativistic corrections appear through  $\varphi$ , the spatial curvature perturbation in the comoving gauge, and these are presented in the right-hand-side of the equations. The background curvature can be regarded as the pure relativistic correction as well.

#### 4.3. Relativistic/Newtonian correspondence

In the flat background, we found that Equations (32)-(33) without  $\varphi$  exactly coincide with the Newtonian hydrodynamic equations of the mass and the momentum conservation as shown in Section 5.3 of [6]. In this section, in order to see the relativistic/Newtonian correspondence in the presence of the background curvature, we consider the weak gravity limit, thus neglect  $\varphi$  terms. Equations (32)-(34) give

$$\dot{\delta} - \kappa = \delta\kappa - \frac{c}{a^2} \chi^i \delta_{,i}, \tag{35}$$

$$\dot{\kappa} + 2H\kappa - \frac{4\pi G}{c^2} \delta\mu = \frac{1}{3} \kappa^2 - \frac{c}{a^2} \chi^i \kappa_{,i} + \frac{c^2}{a^4} \left[ \chi^{i[j} \chi_{,i|j} - \frac{1}{3} (\Delta\chi)^2 \right], \tag{36}$$

$$\kappa + c \frac{\Delta + 3\overline{K}}{a^2} \chi = 0. \tag{37}$$

We identify  $\delta$  and  $\mathbf{u}$  as the Newtonian density and velocity perturbations with

$$\kappa \equiv -\frac{1}{a} \nabla \cdot \mathbf{u} \equiv -\frac{\Delta}{a} u, \tag{38}$$

where  $\mathbf{u} \equiv \nabla u$ . Equation (37) can be arranged to give

$$\frac{\Delta}{a} \left( u - c \frac{\chi}{a} \right) = 3c \frac{\overline{K}}{a^2} \chi. \tag{39}$$

We can rewrite the equations (35), (36) as

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} + \frac{1}{a} \nabla \cdot (\delta \mathbf{u}) = \frac{1}{a} (\nabla \delta) \cdot \nabla \left( u - \frac{c\chi}{a} \right), \tag{40}$$

$$\begin{aligned}
& \frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G \rho \delta + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \\
& = \frac{1}{a^2} \left( u - \frac{c\chi}{a} \right)^{[ij} \left( u + \frac{c\chi}{a} \right)_{|ij} + \frac{2\overline{K}}{a^2} u^{[i} u_{,i} \\
& - \frac{1}{3} \frac{1}{a^2} \left[ \Delta \left( u - \frac{c\chi}{a} \right) \right] \Delta \left( u + \frac{c\chi}{a} \right) + \frac{1}{a^2} \left( u - \frac{c\chi}{a} \right)^{[i} \left[ \Delta \left( u + \frac{c\chi}{a} \right) \right]_{|i}, \tag{41}
\end{aligned}$$

where we have located the background curvature contribution in right-hand-side of the equations. In the presence of the background curvature, we have additional relativistic contribution from the background curvature to the Newtonian equations. Notice that the right-hand-sides are the second order.

In the case of the flat background, the relativistic and Newtonian correspondence was presented in [6]. Up to the second order, the perturbation equations including the



background curvature have been investigated in our previous work [14]. The background curvature effect also appears from the second order in perturbation.

The pure Einstein's gravity contributions appear in terms of the background curvature and  $\varphi$  to the fully nonlinear order. We have shown that the  $\varphi$  terms start to appear from the third order perturbation, thus in a flat background we have exact relativistic/Newtonian correspondences of the density and velocity perturbations to the second-order perturbation [5].

## 5. Minimally coupled scalar field

In this section we present the fully nonlinear and exact formulation for a minimally coupled scalar field.

We consider a minimally coupled scalar field  $\tilde{\phi}$ . The energy and momentum tensor is

$$\tilde{T}_{ab} = \tilde{\phi}_{,a}\tilde{\phi}_{,b} - \left[ \frac{1}{2}\tilde{\phi}^{;c}\tilde{\phi}_{,c} + \tilde{V}(\tilde{\phi}) \right] \tilde{g}_{ab}, \quad (42)$$

and the equation of motion is

$$\tilde{\phi}^{;c}{}_{;c} = \tilde{V}_{,\tilde{\phi}}. \quad (43)$$

Equation (8) gives the fluid quantities. In this section, we set  $c \equiv 1$ .

Since the minimally coupled scalar field has no anisotropic stress, the same fluid formulation remains valid: the fluid quantities are replaced by the ones in terms of the scalar field.

From Equations (7) and (42) we have

$$\tilde{u}_i = -\frac{1}{\tilde{\mu}}\tilde{T}_{ib}\tilde{u}^b = -\frac{\tilde{\phi}_{,i}}{\tilde{\phi}_{,c}\tilde{u}^c}, \quad \tilde{h}_a^b\tilde{\phi}_{,b} = 0, \quad 0 = \tilde{h}^{cd}\tilde{\phi}_{,c}\tilde{\phi}_{,d} = \tilde{\phi}^{;c}\tilde{\phi}_{,c} + \tilde{\phi}^2, \quad (44)$$

where  $\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b$  is the projection tensor and  $\tilde{\phi} \equiv \tilde{\phi}_{,c}\tilde{u}^c$ . We set  $\tilde{\phi} = \phi + \delta\phi$ , where  $\phi(\eta)$  is the background order scalar field, and the  $\delta\phi$  is the perturbed part with arbitrary amplitude.

We obtain

$$\tilde{\phi} = \tilde{\phi}_{,c}\tilde{u}^c = \frac{\tilde{\phi}_{,i}\hat{\gamma}\hat{v}^i}{a(1+2\varphi)} + \hat{\gamma}\frac{D\tilde{\phi}}{Dt} = \frac{1}{\hat{\gamma}}\frac{D\tilde{\phi}}{Dt}, \quad (45)$$

where we define

$$\frac{D}{Dt}\tilde{\phi} \equiv \frac{1}{\mathcal{N}}\left(\frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\tilde{\phi}. \quad (46)$$

From equations (42) and (8) we have

$$\tilde{\mu} = \frac{1}{2}\tilde{\phi}^2 + \tilde{V}, \quad \tilde{p} = \frac{1}{2}\tilde{\phi}^2 - \tilde{V}, \quad a\hat{\gamma}\hat{v}_i\tilde{\phi} = -\tilde{\phi}_{,i}. \quad (47)$$

Equations (9) and (47) provide

$$\hat{v} = \frac{1}{a}\Delta^{-1}\left(\tilde{\phi}^{[k}/\left(\frac{D\tilde{\phi}}{Dt}\right)\right)_{|k}, \quad \hat{v}_i^{(v)} = -\frac{1}{a}\tilde{\phi}_{,i}/\left(\frac{D\tilde{\phi}}{Dt}\right) + \frac{1}{a}\left[\Delta^{-1}\left(\tilde{\phi}^{[k}/\left(\frac{D\tilde{\phi}}{Dt}\right)\right)_{,k}\right]_{,i}. \quad (48)$$

We also have

$$\hat{\gamma} = \frac{1}{\sqrt{1 - \frac{\tilde{\phi}^{[k}\tilde{\phi}_{,k}}{a^2(1+2\varphi)(D\tilde{\phi}/Dt)^2}}}. \quad (49)$$

From equation (43) we have

$$-\tilde{\phi}^{;c}_{;c} = \frac{D^2\tilde{\phi}}{Dt^2} + (3H - \kappa) \frac{D\tilde{\phi}}{Dt} - \frac{(\mathcal{N}\sqrt{1+2\varphi}\tilde{\phi}^{[i})_{|i}}{a^2\mathcal{N}(1+2\varphi)^{3/2}} = -\tilde{V}_{,\tilde{\phi}}(\tilde{\phi}). \quad (50)$$

Equation (50) gives

$$\begin{aligned} \ddot{\tilde{\phi}} + \left( 3H\mathcal{N} - \mathcal{N}\kappa - \frac{\dot{\mathcal{N}}}{\mathcal{N}} - \frac{\chi^i\mathcal{N}_{|i}}{a^2\mathcal{N}(1+2\varphi)} \right) \dot{\tilde{\phi}} + \frac{2\chi^i}{a^2(1+2\varphi)} \dot{\tilde{\phi}}_{,i} \\ - \frac{1}{a^2(1+2\varphi)} \left( \mathcal{N}^2 g^{(3)ij} - \frac{\chi^i\chi^j}{a^2(1+2\varphi)} \right) \tilde{\phi}_{,i|j} + \left[ -\frac{\mathcal{N}^2}{a^2(1+2\varphi)} \left( \frac{\mathcal{N}^{[i}}{\mathcal{N}} + \frac{\varphi^{[i}}{1+2\varphi} \right) \right. \\ \left. + \left( 3H\mathcal{N} - \mathcal{N}\kappa - \frac{\dot{\mathcal{N}}}{\mathcal{N}} - \frac{\chi^k\mathcal{N}_{|k}}{a^2\mathcal{N}(1+2\varphi)} \right) \frac{\chi^i}{a^2(1+2\varphi)} \right. \\ \left. + \left( \frac{\chi^i}{a^2(1+2\varphi)} \right) \cdot + \frac{\chi^k}{a^4(1+2\varphi)} \left( \frac{\chi^i}{1+2\varphi} \right)_{|k} \right] \tilde{\phi}_{,i} \\ = -\mathcal{N}^2 \tilde{V}_{,\tilde{\phi}}(\tilde{\phi}). \end{aligned} \quad (51)$$

To the background order, we have

$$\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \quad (52)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \quad (53)$$

The entropic perturbation  $e$  is given by

$$e \equiv \delta p - \frac{\dot{p}}{\dot{\mu}} \delta \mu = \left( 1 + \frac{\ddot{\phi}}{3H\dot{\phi}} \right) \left( \tilde{\phi}^2 - \dot{\phi}^2 \right) + \frac{2\ddot{\phi}}{3H\dot{\phi}} (\tilde{V} - V). \quad (54)$$

From equation (48) we notice that  $\delta\phi = 0$  implies  $\hat{v}_i = 0$ , and  $\hat{v} = 0$  also implies  $\tilde{\phi} = \text{const}$  in space, thus  $\delta\phi = 0$  as well. Thus, to the fully nonlinear order, the comoving gauge ( $v \equiv 0$ ) implies the uniform-field gauge ( $\delta\phi \equiv 0$ ) and *vice versa*.

In the comoving gauge we have

$$\tilde{\mu} = \frac{1}{2\mathcal{N}^2}\dot{\phi}^2 + V, \quad \tilde{p} = \frac{1}{2\mathcal{N}^2}\dot{\phi}^2 - V. \quad (55)$$

To the background order, we obtain

$$\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \quad (56)$$

and to the fully nonlinear order, we obtain

$$\delta p = \delta \mu = \frac{\dot{\phi}^2}{2} \left( \frac{1}{\mathcal{N}^2} - 1 \right). \quad (57)$$

Now, we see that in the comoving gauge the ideal fluid equations in equations (10)-(18) remain valid with the perturbed equation of state given as  $\delta p = \delta \mu$ .

## 6. Discussion

In this work, we have extended our recent work on the fully nonlinear cosmological perturbation [6] to include the background curvature. The equations are presented in a gauge-ready form, so the temporal gauge conditions are not fixed and can be applied easily depending on the problem to solve. The background curvature terms appear explicitly only in the energy and momentum constraint equations. As an application we considered a zero-pressure irrotational fluid in the comoving gauge. We show that in the presence of the background curvature the pure general relativistic correction appears from the second order. Also, we present the exact and fully nonlinear equations for the minimally coupled scalar field in the non-flat background.

## Acknowledgments

We wish to thank J. Hwang for useful and important suggestions. H.N. was supported by grant No. 2012 R1A1A2038497 from NRF.

## References

- [1] J.M. Bardeen, Particle Physics and Cosmology, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988).
- [2] J. Hwang, *Atsrophys. J.* **375**, 443 (1991).
- [3] J. Hwang and H. Noh, *Phys. Rev. D*, **65**, 023512 (2001).
- [4] J. Hwang and H. Noh, *Phys. Rev. D*, **72**, 044012 (2005).
- [5] H. Noh and J. Hwang, *Phys. Rev. D*, **69**, 104011 (2004).
- [6] J. Hwang and H. Noh, *Mon. Not. R. Astron. Soc.* **433**, 3472 (2013).
- [7] J.W. York, *J. Math. Phys.*, **14**, 456 (1973).
- [8] J. Ehlers, *Proceedings of the mathematical-natural science of the Mainz academy of science and literature*, Nr. **11**, 792 (1961); English translation, *Gen. Rel. Grav.* **25**, 1225 (1993).
- [9] G.F.R. Ellis, *General relativity and cosmology*, *Proceedings of the international summer school of physics Enrico Fermi course 47*, edited by R.K. Sachs, (Academic Press, New York, 1971).
- [10] G.F.R. Ellis, *Cargese Lectures in Physics*, edited by E. Schatzmann, (Gorden and Breach, New York, 1973).
- [11] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, (Cambridge University Press, Cambridge, 1973).
- [12] R. Arnowitt, S. Deser and C.W. Misner, *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York, 1962).
- [13] J. Hwang and H. Noh, *Phys. Rev. D*, **73**, 044021 (2006).
- [14] J. Hwang and H. Noh, *Phys. Rev. D*, **76**, 103527 (2007).

## Appendix: Quantities useful for derivation of fully nonlinear perturbations

Here we present the useful quantities for deriving the fully nonlinear perturbation equations. The ADM and the covariant equations are presented in the Appendices A and C of [6]. Our metric is given by

$$\tilde{g}_{00} = -a^2 (1 + 2\alpha), \quad \tilde{g}_{0i} = -a\chi_i, \quad \tilde{g}_{ij} = a^2 (1 + 2\varphi) g_{ij}^{(3)}. \quad (58)$$

The scale factor  $a$  is a function of conformal time ( $x^0 = \eta$ ), whereas  $\alpha$ ,  $\chi_i$  and  $\varphi$  are general functions of space and time with arbitrary amplitude. The inverse metric is given by

$$\begin{aligned}\tilde{g}^{00} &= -\frac{1}{a^2} \frac{1+2\varphi}{(1+2\varphi)(1+2\alpha) + \chi^k \chi_k / a^2}, \\ \tilde{g}^{0i} &= -\frac{1}{a^2} \frac{\chi^i / a}{(1+2\varphi)(1+2\alpha) + \chi^k \chi_k / a^2}, \\ \tilde{g}^{ij} &= \frac{1}{a^2(1+2\varphi)} \left( g^{(3)ij} - \frac{\chi^i \chi^j / a^2}{(1+2\varphi)(1+2\alpha) + \chi^k \chi_k / a^2} \right).\end{aligned}\quad (59)$$

The ADM metric can be obtained by

$$\begin{aligned}N &= a \sqrt{1+2\alpha + \frac{\chi^k \chi_k}{a^2(1+2\varphi)}} \equiv a\mathcal{N}, \quad N_i = -a\chi_i, \quad N^i = -\frac{\chi^i}{a(1+2\varphi)}, \\ h_{ij} &= a^2(1+2\varphi) g_{ij}^{(3)}, \quad h^{ij} = \frac{1}{a^2(1+2\varphi)} g^{(3)ij},\end{aligned}\quad (60)$$

thus

$$\begin{aligned}\tilde{g}^{00} &= -\frac{1}{a^2 \mathcal{N}^2}, \quad \tilde{g}^{0i} = -\frac{\chi^i}{a^3 \mathcal{N}^2 (1+2\varphi)}, \\ \tilde{g}^{ij} &= \frac{1}{a^2(1+2\varphi)} \left( g^{(3)ij} - \frac{\chi^i \chi^j}{a^2 \mathcal{N}^2 (1+2\varphi)} \right),\end{aligned}\quad (61)$$

and

$$\tilde{n}_i \equiv 0, \quad \tilde{n}_0 = -a\mathcal{N}, \quad \tilde{n}^i = \frac{\chi^i}{a^2 \mathcal{N} (1+2\varphi)}, \quad \tilde{n}^0 = \frac{1}{a\mathcal{N}}. \quad (62)$$

The intrinsic three-space connection and curvatures are given by

$$\begin{aligned}\Gamma^{(h)i}_{jk} &= \Gamma^{(3)i}_{jk} + \frac{1}{1+2\varphi} \left( \varphi_{,j} \delta_k^i + \varphi_{,k} \delta_j^i - \varphi^{[i} g_{jk}^{(3)} \right), \\ \Gamma^{(h)k}_{ik} &= \Gamma^{(3)k}_{ik} + \frac{3\varphi_{,i}}{1+2\varphi}, \\ R^{(h)i}_{jkl} &= R^{(3)i}_{jkl} + \frac{1}{1+2\varphi} \left( \varphi_{,j|k} \delta_\ell^i - \varphi^{[i}_{\phantom{i}k} g_{j\ell}^{(3)} - \varphi_{,j|\ell} \delta_k^i + \varphi^{[i}_{\phantom{i}\ell} g_{jk}^{(3)} \right) \\ &\quad + \frac{1}{(1+2\varphi)^2} \left[ -3\varphi_{,k} \varphi_{,j} \delta_\ell^i + 3\varphi^{[i}_{\phantom{i}k} \varphi_{,k} g_{j\ell}^{(3)} + 3\varphi_{,j} \varphi_{,\ell} \delta_k^i - 3\varphi^{[i}_{\phantom{i}\ell} \varphi_{,\ell} g_{jk}^{(3)} \right. \\ &\quad \left. + \varphi^{[m}_{\phantom{m}m} \varphi_{,m} \left( -\delta_k^i g_{j\ell}^{(3)} + \delta_\ell^i g_{jk}^{(3)} \right) \right], \\ R_{ij}^{(h)} &= R_{ij}^{(3)} - \frac{\varphi_{,i|j}}{1+2\varphi} + 3 \frac{\varphi_{,i} \varphi_{,j}}{(1+2\varphi)^2} - \left( \frac{\varphi^{[k}_{\phantom{k}k}}{1+2\varphi} - \frac{\varphi^{[k}_{\phantom{k}k} \varphi_{,k}}{(1+2\varphi)^2} \right) g_{ij}^{(3)}, \\ R^{(h)} &= \frac{2}{a^2(1+2\varphi)^2} \left( -2\varphi^{[k}_{\phantom{k}k} + 3 \frac{\varphi^{[k}_{\phantom{k}k} \varphi_{,k}}{1+2\varphi} \right) + \frac{1}{a^2(1+2\varphi)} R^{(3)}, \\ \overline{R}^{(h)i}_{\phantom{i}j} &= \frac{1}{a^2(1+2\varphi)^2} \left[ -\varphi^{[i}_{\phantom{i}j} + 3 \frac{\varphi^{[i}_{\phantom{i}j} \varphi_{,j}}{1+2\varphi} - \frac{1}{3} \delta_j^i \left( -\varphi^{[k}_{\phantom{k}k} + 3 \frac{\varphi^{[k}_{\phantom{k}k} \varphi_{,k}}{1+2\varphi} \right) \right],\end{aligned}\quad (63)$$

with

$$R^{(3)i}_{jkl} = \frac{1}{6}R^{(3)}\left(\delta_k^i g_{jl}^{(3)} - \delta_l^i g_{jk}^{(3)}\right), \quad R_{ij}^{(3)} = \frac{1}{3}R^{(3)}g_{ij}^{(3)}, \quad R^{(3)} = 6\bar{K}. \quad (64)$$

It is convenient to have

$$B^i_{|jk} = B^i_{|kj} - R^{(3)i}_{\ell jk}B^\ell, \quad B_{i|jk} = B_{i|kj} + R^{(3)\ell}_{ijk}B_\ell. \quad (65)$$

The extrinsic curvature gives

$$\begin{aligned} K_{ij} &= -\frac{a^2}{\mathcal{N}}\left[(H + \dot{\varphi} + 2H\varphi)g_{ij}^{(3)} + \frac{1}{2a^2}(\chi_{i|j} + \chi_{j|i})\right. \\ &\quad \left. - \frac{1}{a^2(1+2\varphi)}(\chi_i\varphi_{,j} + \chi_j\varphi_{,i} - \chi^k\varphi_{,k}g_{ij}^{(3)})\right], \\ K &= -\frac{1}{\mathcal{N}(1+2\varphi)}\left[3(H + \dot{\varphi} + 2H\varphi) + \frac{1}{a^2}\chi^k_{|k} + \frac{\chi^k\varphi_{,k}}{a^2(1+2\varphi)}\right] \\ &\equiv -3H + \kappa, \\ \bar{K}_j^i &\equiv K_j^i - \frac{1}{3}\delta_j^i K = -\frac{1}{a^2\mathcal{N}(1+2\varphi)}\left[\frac{1}{2}(\chi^i_{|j} + \chi_j^{|i}) - \frac{1}{3}\delta_j^i\chi^k_{|k}\right. \\ &\quad \left. - \frac{1}{1+2\varphi}(\chi^i\varphi_{,j} + \chi_j\varphi_{,i} - \frac{2}{3}\delta_j^i\chi^k\varphi_{,k})\right]. \end{aligned} \quad (66)$$

The fluid four-vector is

$$\begin{aligned} \tilde{u}_i &\equiv \frac{a\hat{\gamma}\hat{v}_i}{c}, \quad \tilde{u}_0 = -a\mathcal{N}\hat{\gamma} - \frac{\chi^k\hat{\gamma}\hat{v}_k}{c(1+2\varphi)}, \\ \tilde{u}^i &= \frac{\hat{\gamma}\hat{v}^i}{ca(1+2\varphi)} + \frac{\hat{\gamma}\chi^i}{a^2\mathcal{N}(1+2\varphi)}, \quad \tilde{u}^0 = \frac{1}{a\mathcal{N}}\hat{\gamma}. \end{aligned} \quad (67)$$

For  $\hat{v}_i = 0$  we have  $\tilde{u}_a = \tilde{n}_a$ . The energy-momentum tensor of an ideal fluid is given by

$$\begin{aligned} \tilde{T}_0^0 &= -\tilde{\mu} - \frac{\tilde{\mu} + \tilde{p}}{1+2\varphi}\left(\frac{\hat{\gamma}^2\hat{v}^i\hat{v}_i}{c^2} + \frac{1}{a\mathcal{N}}\chi^i\frac{\hat{\gamma}^2\hat{v}_i}{c}\right), \quad \tilde{T}_i^0 = \frac{1}{\mathcal{N}}(\tilde{\mu} + \tilde{p})\frac{\hat{\gamma}^2\hat{v}_i}{c}, \\ \tilde{T}_{ij} &= a^2\left[(1+2\varphi)\tilde{p}g_{ij}^{(3)} + (\tilde{\mu} + \tilde{p})\frac{\hat{\gamma}^2\hat{v}_i\hat{v}_j}{c^2}\right]. \end{aligned} \quad (68)$$

The ADM fluid quantities are

$$\begin{aligned} E &= \tilde{\mu} + (\tilde{\mu} + \tilde{p})(\hat{\gamma}^2 - 1), \quad J_i = a(\tilde{\mu} + \tilde{p})\frac{\hat{\gamma}^2\hat{v}_i}{c}, \quad J^i = \frac{(\tilde{\mu} + \tilde{p})}{a(1+2\varphi)}\frac{\hat{\gamma}^2\hat{v}^i}{c}, \\ S_j^i &= \tilde{p}\delta_j^i + \frac{(\tilde{\mu} + \tilde{p})}{(1+2\varphi)}\frac{\hat{\gamma}^2\hat{v}^i\hat{v}_j}{c^2}, \quad S = 3\tilde{p} + (\tilde{\mu} + \tilde{p})(\hat{\gamma}^2 - 1), \\ \bar{S}_j^i &= (\tilde{\mu} + \tilde{p})\left[\frac{\hat{\gamma}^2}{1+2\varphi}\frac{\hat{v}^i\hat{v}_j}{c^2} - \frac{1}{3}\delta_j^i(\hat{\gamma}^2 - 1)\right]. \end{aligned} \quad (69)$$